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A Frequency-Domain Criterion for the Quadratic Stability of Discrete-Time Systems with Switching between Three Linear Subsystems

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Abstract—Connected systems with switching between three linear discrete-time subsystems are considered, and a new frequency-domain criterion for the existence of a quadratic Lyapunov function ensuring the stability of such systems under arbitrary switching is proposed. The application of this criterion is demonstrated on an example of a third-order system.

Keywords: discrete-time switched systems, stability, Lyapunov functions, matrix inequalities

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1. INTRODUCTION

The theory of discrete-time systems has been actively developing lately. Various aspects of this theory have been discussed in relatively recent publications [1–7]; also, see the bibliography therein. This paper is devoted to the quadratic stability problem of connected discrete-time systems [3] with switching between three linear stationary subsystems under any switching laws. The term "connected system" will be explained below. By quadratic stability we mean the stability of a system that can be established using a Lyapunov function from the class of quadratic forms or quadratic Lyapunov functions (QLFs). For a connected system with switching between two subsystems, this problem is equivalent to the absolute stability problem of a discrete-time system with a single nonlinearity [3], and a quadratic stability criterion for such a system is the well-known Tsypkin's criterion [8]. In the case of switching between two subsystems, connectedness means that the rank of the difference of the matrices determining the switched subsystems is one.

For connected discrete-time systems with switching between three linear subsystems, a frequency-domain criterion for the existence of a QLF was established in [3]. The disadvantages are an excessively cumbersome procedure for obtaining this criterion and an excessively cumbersome form of the final result. They can be explained as follows. The quadratic stability of a switched system ensues from the existence of a common quadratic Lyapunov function (CQLF). In the case under consideration, the existence of a CQLF is determined by the feasibility of a system of three Lyapunov linear matrix inequalities (LMIs) for discrete-time systems. This system of LMIs is connected, and one resulting matrix inequality equivalent to it was derived in [3]. However, (a) this matrix inequality is not an LMI and (b) the frequency-domain conditions of its feasibility cannot be obtained based on the generalized Kalman–Szegö–Popov lemma [9, 10], as it was done in [3] in the case of Tsypkin's criterion. To overcome inconvenience (b), a fractional linear transformation was used in [3] to pass from the system of LMIs for discrete-time systems to the equivalent system of Lyapunov LMIs for continuous-time systems. The resulting matrix inequality for this system is again not an LMI, but its feasibility conditions were established in [3] in the form of a frequency-domain criterion based on the frequency theorem [11, p. 54] (the Kalman–Yakubovich–Popov (KYP) lemma). The conditions of this criterion are expressed through the elements of a "transfer matrix" for the continuous-time system obtained by the transformation. Finally, using a rather cumbersome procedure, these elements are expressed through the elements of the "transfer matrix" of the original discrete-time system.

In this paper, we apply a new result (Theorem 2 of [12]) to the original system of three Lyapunov LMIs for discrete-time systems to obtain an equivalent resulting matrix inequality that is an LMI. Next, we demonstrate that the feasibility of this LMI can be established by the generalized Kalman–Szegö–Popov lemma in the frequency-domain criterion form. This yields a new frequency-domain criterion for the quadratic stability of the systems under consideration, the main aim of the paper.

Section 2 describes the system of three Lyapunov LMIs for discrete-time systems, whose feasibility is equivalent to the quadratic stability of the systems under consideration. The main result of this paper—the frequency-domain criterion for quadratic stability—is presented in Section 3. A numerical example of a third-order system is provided in Section 4; for this system, the proposed criterion is applied to analytically find the entire quadratic stability domain on the parameter.

2. PROBLEM STATEMENT

Consider a linear discrete-time switched system of the form

$$x(t+1) = A(t)x(t), \quad A(t) \in \overline{A} = \{A_1, A_2, A_3\}, \tag{1}$$

where $A_s \in \mathbb{R}^{n\times n}$ and $A(t) : \mathrm{Z}_+ \longrightarrow \overline{A}$ is a mapping from the set Z_+ of nonnegative integers into $\overline{A}$. By assumption, the matrices A_s are stable (Schur, see [13]), i.e., $r(A_s) = \max\limits_{\nu} |\mu_\nu(A_s)| < 1$ for $s = \overline{1,3}$, where μ_ν denote the eigenvalues of the matrix A_s. The stability of the switched system (1) will be analyzed using QLFs of the form

$$v(x) = x^\top Lx, \quad L = L^\top = \|l_{ij}\|_{i,j=1}^n, \tag{2}$$

where the symbol $\{\cdot\}^\top$ means transpose.

According to [3], the existence of a QLF (2) is determined by the feasibility of the system of LMIs

$$I_s = A_s^\top LA_s - L < 0, \quad s = \overline{1,3}. \tag{3}$$

System (1) is connected [3] if the matrices $\{A_1, A_2, A_3\}$ can be represented as

$$\begin{aligned} A_1 &= A, \\ A_2 &= A + b_1c_1^\top, \qquad b_i, c_i \in \mathbb{R}^n. \\ A_3 &= A + b_2c_2^\top, \end{aligned} \tag{4}$$

In this case, system (3) can be written in the form

$$\begin{aligned} I_1 &= A^\top LA - L < 0, \\ I_2 &= (A + b_1c_1^\top)^\top L(A + b_1c_1^\top) - L < 0, \\ I_3 &= (A + b_2c_2^\top)^\top L(A + b_2c_2^\top) - L < 0. \end{aligned} \tag{5}$$

The problem under consideration is to obtain a frequency-domain criterion for the feasibility of the system of LMIs (5).

3. SYSTEMS WITH SWITCHING BETWEEN THREE LINEAR DISCRETE-TIME SUBSYSTEMS

To investigate the feasibility of system (5) we use Theorem 2 of [12]. In the formulas below, the symbols "•" denote the elements below the principal diagonal of an appropriate symmetric matrix that coincide with the corresponding elements above this diagonal.

Theorem 1. *Let the inequalities in the system*

$$I_1 < 0, \quad I_2 = I_1 + Q_1 < 0, \quad I_3 = I_1 + Q_2 < 0 \tag{6}$$

be LMIs with respect to the unknown variable ν, *i.e.,* $I_s = I_s(\nu)$, $s = \overline{1,3}$, *and* $Q_j(\nu) = p_j(\nu)q_j^\top + q_j p_j^\top(\nu)$, *where* $p_j = p_j(\nu)$ *linearly depends on* ν *and* q_j *is independent of* ν, $j = 1, 2$. *Then system* (6) *is equivalent to the single matrix inequality*

$$\hat{\hat{I}} = \begin{pmatrix} I_1(\nu) & p_1(\nu) + \dfrac{\tau_1}{2}q_1 & p_2(\nu) - p_1(\nu) + \dfrac{\tau_2}{2}q_2 - \dfrac{\tau_1}{2}q_1 \\ (\bullet)^\top & -\tau_1 & \dfrac{\tau_1 - \tau_2 + \tau_3}{2} \\ (\bullet)^\top & \bullet & -\tau_3 \end{pmatrix} < 0, \tag{7}$$

which is an LMI with respect to $(\nu, \tau_1, \tau_2, \tau_3)$.

With Theorem 1 applied to system (5), the feasibility of system (5) becomes equivalent to the feasibility of the single matrix inequality with respect to the elements of the matrix L and the three additional parameters τ_1, τ_2, τ_3. The applicability of Theorem 1 to system (5) and the resulting matrix inequality follow from the relations below. Let the matrix $I_1(\nu)$ be the matrix $(A^\top LA - L)$ of system (5), i.e., $I_1(\nu) = I_1(L) = A^\top LA - L$. (The role of the parameter ν is played by the matrix L.) The difference of the matrices $(I_2 - I_1)$ from (5) can be represented as $p_1q_1^\top + q_1p_1^\top$:

$$\begin{aligned} I_2 - I_1 = A_2^\top LA_2 - A_1^\top LA_1 &= (A + b_1c_1^\top)^\top L(A + b_1c_1^\top) - A^\top LA \\ &= (A^\top L + c_1b_1^\top L)(A + b_1c_1^\top) - A^\top LA \\ &= A^\top Lb_1c_1^\top + c_1b_1^\top LA + c_1b_1^\top Lb_1c_1^\top. \end{aligned} \tag{8}$$

With the notations $p_1^0 = p_1^0(L) = A^\top Lb_1$ and $\delta_{11} = \delta_{11}(L) = b_1^\top Lb_1$, we have

$$I_2 - I_1 = p_1^0c_1^\top + c_1(p_1^0)^\top + \delta_{11}c_1c_1^\top = p_1q_1^\top + q_1p_1^\top, \tag{9}$$

where $p_1 = p_1(L) = A^\top Lb_1 + \left(\frac{\delta_{11}(L)}{2}\right)c_1$ and $q_1 = c_1$.

Similarly, let $p_2^0 = p_2^0(L) = A^\top Lb_2$ and $\delta_{22} = \delta_{22}(L) = b_2^\top Lb_2$; then

$$I_3 - I_1 = p_2^0c_2^\top + c_2(p_2^0)^\top + \delta_{22}c_2c_2^\top = p_2q_2^\top + q_2p_2^\top, \tag{10}$$

where $p_2 = p_2(L) = A^\top Lb_2 + \left(\frac{\delta_{22}(L)}{2}\right)c_2$ and $q_2 = c_2$.

Thus, by Theorem 1, system (5) is equivalent to the single matrix inequality

$$\hat{\hat{I}} = \begin{pmatrix} A^\top LA - L & p_1(L) + \dfrac{\tau_1}{2}c_1 & p_2(L) - p_1(L) + \dfrac{\tau_2}{2}c_2 - \dfrac{\tau_1}{2}c_1 \\ (\bullet)^\top & -\tau_1 & \dfrac{\tau_1 - \tau_2 + \tau_3}{2} \\ (\bullet)^\top & \bullet & -\tau_3 \end{pmatrix} < 0, \tag{11}$$

which is an LMI with respect to $(L, \tau_1, \tau_2, \tau_3)$.

Now we demonstrate that the feasibility of the LMI (11) is determined based on the generalized Kalman–Szegö–Popov lemma [10].

Lemma 1. *The LMI* (11) *is equivalent to the LMI*

$$\begin{pmatrix} A^\top LA - L & A^\top L\widehat{B} + \dfrac{\widehat{C}\tau}{2} \\ \widehat{B}^\top LA + \dfrac{\tau\widehat{C}^\top}{2} & \widehat{B}^\top L\widehat{B} - \Gamma \end{pmatrix} < 0, \tag{12}$$

where

$$\widehat{B} = \begin{pmatrix} \widehat{B}_1 & \widehat{B}_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 - b_1 \end{pmatrix}, \quad \widehat{C} = \begin{pmatrix} \widehat{C}_1 & \widehat{C}_2 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 - \dfrac{\widehat{\tau}_1}{\widehat{\tau}_2} c_1 \end{pmatrix},$$

$$\tau = \begin{pmatrix} \widehat{\tau}_1 & 0 \\ 0 & \widehat{\tau}_2 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \widehat{\tau}_1 & \dfrac{-\widehat{\tau}_1 + \widehat{\tau}_2 - \widehat{\tau}_3}{2} \\ \bullet & \widehat{\tau}_3 \end{pmatrix}.$$

The proof of Lemma 1 is given in the Appendix.

Necessary and sufficient conditions for the feasibility of the LMI (12) are determined in the form of a frequency-domain inequality from the generalized Kalman–Szegö–Popov lemma [9, 10]. As a result, we arrive at the following quadratic stability criterion for system (1).

Theorem 2. *Let the matrix A be Schur* ($r(A) < 1$), *and let there exist numbers $\widehat{\tau}_s > 0$, $s = \overline{1,3}$, such that $\Gamma > 0$ and the frequency-domain inequality*

$$D(\lambda) = \Gamma + \mathrm{Re}\,[\tau\widehat{C}^\top (A - \lambda E_n)^{-1}\widehat{B}] > 0 \tag{13}$$

holds for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$, where E_n is an identity matrix of dimensions $(n \times n)$. (In this inequality, $\mathrm{Re}\,W = (W + W^*)/2$, $W^* = \overline{W}^\top$ is the Hermitian conjugate to W; from this point onwards, the symbol $\overline{\{\cdot\}}$ means complex conjugation and the inequality sign is interpreted as the positive definiteness of an appropriate Hermitian form.) Then the connected system (1) *has a CQLF (system* (5) *is feasible, and system* (1) *is stable). If system* (5) *feasible, then such a set of numbers $\widehat{\tau}_s > 0$, $s = \overline{1,3}$, exists.*

Let us write the frequency-domain condition (13) in detail. It seems logical to treat $W(p) = C^\top(A - pE_n)^{-1}B$, $p \in \mathbb{C}$, as an analog of the transfer matrix for system (1), where $C = \begin{pmatrix} c_1 & c_2 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$. With the notation $\Delta(p) = (A - pE_n)^{-1}$, we have

$$W(p) = C^\top\Delta(p)B = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, \quad \text{where} \quad w_{ij}(p) = c_i^\top\Delta(p)b_j. \tag{14}$$

For the sake of simplicity, we eliminate the hats, using τ_s instead of $\widehat{\tau}_s$. From (13) it follows that

$$D(\lambda) = \Gamma + \mathrm{Re}\,\tau\widehat{W}(\lambda) = \Gamma + 1/2\left[\tau\widehat{W}(\lambda) + \widehat{W}^*(\lambda)\tau^\top\right],$$

where

$$\widehat{W}(\lambda) = \widehat{C}^\top\Delta(\lambda)\widehat{B}\begin{pmatrix} c_1 & c_2 - \dfrac{\tau_1}{\tau_2}c_1 \end{pmatrix}^\top \Delta(\lambda)\begin{pmatrix} b_1 & b_2 - b_1 \end{pmatrix}$$

$$= \begin{pmatrix} w_{11}(\lambda) & w_{12}(\lambda) - w_{11}(\lambda) \\ w_{21}(\lambda) - \dfrac{\tau_1}{\tau_2}w_{11}(\lambda) & w_{22}(\lambda) - \dfrac{\tau_1}{\tau_2}w_{12}(\lambda) - w_{21}(\lambda) + \dfrac{\tau_1}{\tau_2}w_{11}(\lambda) \end{pmatrix}.$$

Finally, we write the inequality $D(\lambda) > 0$ from (13) as

$$D(\lambda) = \Gamma + \frac{1}{2}\begin{pmatrix} 2\tau_1 \mathrm{Re}\, w_{11} & \tau_1 w_{12} + \tau_2 \overline{w_{21}} - 2\tau_1 \mathrm{Re}\, w_{11} \\ \overline{(\bullet)} & 2\tau_1 \mathrm{Re}\,(w_{11} - w_{12}) + 2\tau_2 \mathrm{Re}\,(w_{22} - w_{21}) \end{pmatrix} > 0. \tag{15}$$

(For the sake of brevity, w_{ij} is taken instead of $w_{ij}(\lambda)$.)

Remark 1. Theorem 2 remains valid when replacing inequality (13) with inequality (15), where $w_{ij} = w_{ij}(\lambda) = c_i^\top \Delta(\lambda) b_j$, $i, j = 1, 2$.

If system (1) is a triangular switched system [3], i.e., $c_1 = c_2 \triangleq c$, then $w_{11} = w_{21} \triangleq W_1 = c^\top \Delta(\lambda) b_1$ and $w_{22} = w_{12} \triangleq W_2 = c^\top \Delta(\lambda) b_2$. In this case, inequality (15) can be written as

$$D(\lambda) = \begin{pmatrix} \tau_1(1 + \mathrm{Re}\, W_1) & \dfrac{-\tau_1 + \tau_2 - \tau_3 + \tau_1 W_2 + \tau_2 \overline{W_1}}{2} - \tau_1 \mathrm{Re}\, W_1 \\ \overline{(\bullet)} & \tau_3 + (\tau_2 - \tau_1)\,(\mathrm{Re}\, W_2 - \mathrm{Re}\, W_1) \end{pmatrix} > 0. \tag{16}$$

Remark 2. For the triangular system (1) ($c_1 = c_2 = c$), Theorem 2 remains valid when replacing inequality (13) with inequality (16), where $W_j = W_j(\lambda) = c^\top \Delta(\lambda) b_j$, $j = 1, 2$.

Compare conditions (15) and (16) of the criterion in Theorem 2 for connected switched systems and triangular switched systems with those of Theorem 2 from [3] and their modification for triangular systems (formulas (6.3)–(6.5) from [3]). Significant progress is evident.

Remark 3. Inequalities (13), (15), and (16) are linear in the parameter $\mathcal{T}$; therefore, without losing generality, let $\tau_3 = 1$ in these inequalities. Thus, the inequalities under consideration will contain only two additional parameters each: $\tau_1 > 0$ and $\tau_2 > 0$.

The well-known Tsypkin's criterion [8] is a quadratic stability criterion under switching between two subsystems. The criterion of Theorem 2 can be considered an analog of Tsypkin's criterion under switching between three subsystems.

4. NUMERICAL SOLUTION

The quadratic stability problem for system (1) is numerically solved by applying standard software tools for checking the feasibility of the system of LMIs (5) of dimension $3n$ with respect to $n(n+1)/2$ unknowns. Due to Lemma 1, it is possible to check the feasibility of the single LMI (12) of dimension $(n+2)$ with respect to $n(n+1)/2 + 3$ unknowns instead of the system of LMIs (5). This transition allows significantly simplifying the problem, especially for large n.

5. AN EXAMPLE

Consider a connected switched system of the form (1) from the example presented in [3]. In this example, the matrices A_s in (1) are given by (4) with

$$A_1 = A = \begin{pmatrix} 0 & 0 & -0.5 \\ 0.5 & 0 & -1.5 \\ 0 & 0.5 & -1.5 \end{pmatrix}, \quad b_1 = k_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad b_2 = k_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$
$$c_1 = c_2 = c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{17}$$

where $k_i \geqslant 0$ are the parameters determining the stability domain of the switched system. Then the matrices A_2 and A_3 take the form

$$A_2 = \begin{pmatrix} 0 & 0 & -0.5 \\ 0.5 & 0 & -1.5 \\ 0 & 0.5 & -1.5 + k_1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & -0.5 \\ 0.5 & 0 & -1.5 + k_2 \\ 0 & 0.5 & -1.5 \end{pmatrix}. \tag{18}$$

In the sequel, system (1) with the matrices A_s (17), (18) will be referred to as system (1;17).

Re-examining the example from [3] can be explained as follows. In the example from [3], given $k_1 = k_2 = k$, the entire quadratic stability domain on the parameter k was found. This result was obtained using the necessary (separately) and sufficient (separately) conditions for the feasibility of the system of LMIs (5). As it turned out, the estimates under these conditions coincide; hence, the resulting quadratic stability domain is entire. Note that the conditions from [3] essentially rest on the triangular property of the system, i.e., $c_1 = c_2 = c$.

This section aims to repeat the result from [3] based on the criterion of Theorem 2. Although the considerations below use a variant of Theorem 2 from Remark 2, this theorem does not include the triangularity requirement.

The presentation here involves the auxiliary calculations from [3]. Obviously, the matrix A_1 is Schur, $|\mu_i(A_1)| < 1$, since $\mu_i(A_1) = -0.5$, $i = \overline{1,3}$. The matrix A_2 is Schur for $k_1 \in [0,\, 3.375)$, whereas the matrix A_3 is Schur for $k_2 \in [0,\, 0.25)$.

The functions $W_j(\lambda) = c^\top (A - \lambda E_n)^{-1} b_j$ from (16) have the form

$$W_1(\lambda) = -8k_1\lambda^2/(2\lambda+1)^3 \ \text{ and } \ W_2(\lambda) = -4k_2\lambda/(2\lambda+1)^3,$$

$\det(A - \lambda E) = -(0.5+\lambda)^3$. Inequality (16) should be checked for all $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. For the set $|\lambda| = 1$, we use the parameterization $\lambda = \frac{1-i\omega}{1+i\omega}$ for all $\omega \in [-\infty, \infty]$. Let us calculate $W_1(\lambda)$ and $W_2(\lambda)$ for $\lambda = \frac{1-i\omega}{1+i\omega}$. We write the real and imaginary parts of $W_j\left(\frac{1-i\omega}{1+i\omega}\right)$, simultaneously adopting the simplified notations $\operatorname{Re} W_j\left(\frac{1-i\omega}{1+i\omega}\right) = R_j(\omega) = R_j$ and $\operatorname{Im} W_j\left(\frac{1-i\omega}{1+i\omega}\right) = I_j(\omega) = I_j$ (see [3]):

$$\begin{aligned} R_1 = R_1(\omega) &= \operatorname{Re} W_1\left(\frac{1-i\omega}{1+i\omega}\right) = \frac{-8k_1(1+\omega^2)(27+18\omega^2-\omega^4)}{(9+\omega^2)^3}, \\ I_1 = I_1(\omega) &= \operatorname{Im} W_1\left(\frac{1-i\omega}{1+i\omega}\right) = \frac{-64k_1\omega^3(1+\omega^2)}{(9+\omega^2)^3}, \\ R_2 = R_2(\omega) &= \operatorname{Re} W_2\left(\frac{1-i\omega}{1+i\omega}\right) = \frac{-4k_2(1+\omega^2)(27-36\omega^2+\omega^4)}{(9+\omega^2)^3}, \\ I_2 = I_2(\omega) &= \operatorname{Im} W_2\left(\frac{1-i\omega}{1+i\omega}\right) = \frac{-4k_2(1+\omega^2)(54\omega-10\omega^3)}{(9+\omega^2)^3}. \end{aligned} \tag{19}$$

In terms of (19), inequality (16) takes the form

$$D(\omega) = \begin{pmatrix} \tau_1(1+R_1) & \dfrac{-\tau_1+\tau_2-\tau_3+\tau_1R_2+\tau_2R_1-2\tau_1R_1}{2} + i\dfrac{\tau_1I_2-\tau_2I_1}{2} \\ \overline{(\bullet)} & \tau_3 + (\tau_2-\tau_1)(R_2-R_1) \end{pmatrix} > 0.$$

Letting $k_2 = k_1 = k$, we make the change $\omega^2 = y \geqslant 0$. It is required to find the largest domain $[0, k^*)$ for which there exists a set of parameters $\tau_i > 0$, $j = 1,2,3$, such that $D(\omega) \cong D(y) > 0$ for $k \in [0, k^*)$ and all $y \geqslant 0$. Checking the inequality $D(y) > 0$ reduces to checking the inequalities

$$\text{(A) } D_{11} = \tau_1(1+R_1) > 0, \quad \text{(B) } D_{22} = \tau_3 + (\tau_2-\tau_1)(R_2-R_1) > 0, \quad \text{(C) } \det D(y) > 0,$$

where $D_{ij} = D_{ij}(y)$, $i, j = 1, 2$, are the elements of the matrix $D(y)$. (In fact, it suffices to check (A) and (C).) Inequality (A) is equivalent to

$$\begin{aligned} P_1(y) &= (9+y)^3 D_{11}(y) = \tau_1(1+R_1) \\ &= \tau_1(9+y)^3 - 8\tau_1 k(1+y)(27+18y-y^2) \\ &= \tau_1(1+8k)y^3 + \tau_1(27-136k)y^2 + \tau_1(243-360k)y + \tau_1 27(27-8k) > 0. \end{aligned}$$

The check of inequality (A) coincides with that of inequalities (7.4) and (7.5) from [3]. As was shown in [3], for $k < 0.44$, the inequality $P_1(y) > 0$ holds for all $y \geqslant 0$.

In view of Remark 3, we assume that $\tau_3 = 1$ and, for brevity, $\tau_2 - \tau_1 \triangleq \delta$. Then checking inequality (B) reduces to checking the inequality

$$\begin{aligned} P_2(y) &= (9+y)^3 D_{22}(y) = (9+y)^3 + 4k\delta(1+y)(27+72y-3y^2) \\ &= (1-12k\delta)y^3 + (27+276k\delta)y^2 + (243+396k\delta)y + 729 + 108k\delta > 0. \end{aligned}$$

Consider inequality (C):

$$\det D = D_{11}D_{22} - D_{12}\overline{D_{12}} = D_{11}D_{22} - (\mathrm{Re}D_{12})^2 - (\mathrm{Im}D_{12})^2 > 0.$$

With the notations $P_3(y) \triangleq 2(9+y)^3 \operatorname{Re} D_{12}$ and $P_4(y) \triangleq 2(9+y)^3 \operatorname{Im} D_{12}$, we have

$$\begin{aligned} P_3(y) &= 2(9+y)^3 \operatorname{Re} D_{12}(y) = 2\big(\tau_1(R_2 - R_1) + \delta R_1 + \delta - 1\big)(9+y)^3, \\ P_4(y) &= 2(9+y)^3 \operatorname{Im} D_{12}(y) = 2\big(\tau_1 I_2 - \tau_2 I_1\big)(9+y)^3. \end{aligned}$$

Using the expressions from (19) gives

$$\begin{aligned} P_3(y) &= y^3[4k(2\delta - 3\tau_1) + \delta - 1] + y^2[27(\delta-1) + 4k(69\tau_1 - 34\delta)] \\ &\quad + y[9(27(\delta-1) - 40k\delta + 44k\tau_1)] + [27(27(\delta-1) - 4k(2\delta - \tau_1))], \\ P_4(y) &= 4k\tau_1\sqrt{y}(1+y)(10y-54) + 64k\tau_2 y\sqrt{y}(1+y) \\ &= 4k\sqrt{y}(1+y)\big(\tau_1(10y-54) + 16\tau_2 y\big). \end{aligned}$$

Inequality (C) is equivalent to

$$P(y) \triangleq (9+y)^6 \det D(y) = P_1(y)P_2(y) - \frac{1}{4}P_3(y)^2 - \frac{1}{4}P_4(y)^2 > 0. \tag{20}$$

The polynomial $P(y)$ is of degree 6 in the variable y. Its coefficients $f_s = f_s(k)$ for y^s are functions of k that depend on the additional parameters τ_1 and τ_2. The coefficient $f_6(k)$ of this polynomial at y^6 is

$$f_6(k) = \tau_1(1+8k)(1-12k\delta) - (1/4)[4k(2\delta - 3\tau_1) + \delta - 1]^2.$$

The condition $f_6(k) \geqslant 0$ is necessary for fulfilling $P(y) > 0$ for all $y \geqslant 0$. The function $f_6(k)$ represents a polynomial of degree 2 in the variable k. Its coefficient at k^2 is $a_6 = -96\tau_1\delta - 4(2\delta - 3\tau_1)^2 = -4(2\tau_2 + \tau_1)^2$, i.e., $a_6 < 0$ since $\tau_j > 0$. It follows that $f_6(k)$ is a concave function. The desired domain $[0, k^*)$ can be estimated from above by the half-interval $[0,\ 0.25)$ (the Schur domain of the matrix A_3). We check the values $f_6(0)$ and $f_6(0.25)$:

$$\begin{aligned} f_6(0) &= \tau_1 - (1/4)(\delta - 1)^2, \\ f_6(0.25) &= \tau_1(1+2)(1-3\delta) - (1/4)\big((2\delta - 3\tau_1) + \delta - 1\big)^2. \end{aligned}$$

The condition $f_6(0) > 0$ gives the parameter estimate $4\tau_1 > (\delta - 1)^2$. Let us transform the expression for $f_6(0.25)$:

$$f_6(0.25) = 3\tau_1(1-3\delta) - \frac{1}{4}(3\delta - 3\tau_1 - 1)^2 = -\frac{1}{4}(3\delta + 3\tau_1 - 1)^2 = -\frac{1}{4}(3\tau_2 - 1)^2.$$

As a result, $f_6(0.25) < 0$ for all parameter values except for $\tau_2 = 1/3$. Thus, letting $\tau_2 = 1/3$ is the single possibility to obtain the largest domain $[0, k^*)$ in which $f_6(0.25) > 0$. If we take $\tau_2 = 1/3$ and define τ_1 so that $f_6(0) > 0$, the concavity of $f_6(k)$ will imply $f_6(k) > 0$ for all $k \in [0,\, 0.25)$. Partly by chance, partly to obtain $\delta = 0$, we set $\tau_1 = \tau_2 = 1/3$. In this case, $f_6(0) = 1/12 > 0$.

As it turns out, for $\tau_1 = \tau_2 = 1/3$, the other coefficients $f_s(k)$, $s = 0, \ldots, 5$, of the polynomial $P(y) = \sum\limits_{s=0}^{6} f_s(k) y^s$ from (20) are concave functions in the variable k. In addition, the inequalities $f_s(0) > 0$ and $f_s(0.25) > 0$, $s = 0, \ldots, 5$, hold for the values of these functions at the limit points of the half-interval $[0,\, 0.25)$. The tedious verification of this fact by elementary algebra techniques is omitted here. Thus, we have $f_s(k) > 0$ for all $k \in [0,\, 0.25)$, $s = 0, \ldots, 6$. Hence, inequality (20) is valid for all $y \geqslant 0$. According to Theorem 2, the quadratic stability domain of system (1;17) is exhausted by the set $[0,\, 0.25)$. Due to its coincidence with the Schur domain of the matrices $\{A_1, A_2, A_3\}$ defining system (1;17) (on the parameter $k_1 = k_2 = k$), this domain is the entire stability domain of system (1;17) under arbitrary switching.

6. CONCLUSIONS

A connected system with switching between three linear discrete-time subsystems has been considered. An existence criterion for a QLF of such systems has been established, both as a frequency-domain condition and as feasibility conditions of a single LMI. As an illustrative example, the frequency-domain criterion has been applied to a third-order system, analytically yielding its entire quadratic stability domain on the parameter k. In the case under study, this domain coincides with the entire stability domain of system (1;17) under arbitrary switching.

APPENDIX

Proof of Lemma 1. We define the new parameters

$$\widehat{\tau}_1 \triangleq \delta_{11} + \tau_1, \quad \widehat{\tau}_2 \triangleq \delta_{22} + \tau_2.$$

Then

$$\begin{aligned} & p_1(L) + \frac{\tau_1}{2} c_1 = A^\top L b_1 + \frac{\delta_{11}}{2} c_1 + \frac{\tau_1}{2} c_1 = A^\top L \widehat{B}_1 + \frac{\widehat{\tau}_1}{2} \widehat{C}_1, \\ & p_2(L) - p_1(L) + \frac{\tau_2}{2} c_2 - \frac{\tau_1}{2} c_1 \\ & = A^\top L b_2 - A^\top L b_1 + \frac{\delta_{22} + \tau_2}{2} c_2 - \frac{\delta_{11} + \tau_1}{2} c_1 \\ & = A^\top L (b_2 - b_1) + \frac{\widehat{\tau}_2}{2} c_2 - \frac{\widehat{\tau}_1}{2} c_1 = A^\top L \widehat{B}_2 + \frac{\widehat{\tau}_2}{2} \widehat{C}_2. \end{aligned} \tag{A.1}$$

It suffices to represent the matrix $\begin{pmatrix} -\tau_1 & \frac{\tau_1 - \tau_2 + \tau_3}{2} \\ \bullet & -\tau_3 \end{pmatrix}$ in the form $\left(\widehat{B}^\top L \widehat{B} - \Gamma\right)$.

Considering $b_1^\top L b_2 \triangleq \delta_{12}$ and $b_2^\top L b_1 \triangleq \delta_{21}$, we write the matrix $\widehat{B}^\top L \widehat{B}$ as

$$\begin{aligned}\widehat{B}^\top L \widehat{B} = \begin{pmatrix} b_1^\top \\ b_2^\top - b_1^\top \end{pmatrix} L \begin{pmatrix} b_1 & b_2 - b_1 \end{pmatrix} = \begin{pmatrix} b_1^\top L \\ b_2^\top L - b_1^\top L \end{pmatrix} \begin{pmatrix} b_1 & b_2 - b_1 \end{pmatrix} \\ = \begin{pmatrix} \delta_{11} & \delta_{12} - \delta_{11} \\ \delta_{21} - \delta_{11} & \delta_{22} - 2\delta_{12} + \delta_{11} \end{pmatrix}.\end{aligned} \tag{A.2}$$

Thus, it is required to find the elements of the matrix $\Gamma = \|\gamma_{ij}\|_{i,j=1}^n$ so that

$$\begin{pmatrix} -\tau_1 & \dfrac{\tau_1 - \tau_2 + \tau_3}{2} \\ \bullet & -\tau_3 \end{pmatrix} = \begin{pmatrix} \delta_{11} - \gamma_{11} & \delta_{12} - \delta_{11} - \gamma_{12} \\ \delta_{21} - \delta_{11} - \gamma_{21} & \delta_{22} - 2\delta_{12} + \delta_{11} - \gamma_{22} \end{pmatrix}. \tag{A.3}$$

Since $-\tau_1 = \delta_{11} - \widehat{\tau}_1$, the equality of the elements $\{\cdot\}_{11}$ of the matrices from (A.3) gives $\gamma_{11} = \widehat{\tau}_1$. In view of $-\tau_2 = \delta_{22} - \widehat{\tau}_2$, the equality of the elements $\{\cdot\}_{12}$ leads to

$$\frac{\tau_1 - \tau_2 + \tau_3}{2} = \frac{-\delta_{11} + \widehat{\tau}_1 + \delta_{22} - \widehat{\tau}_2 + \tau_3}{2} = \delta_{12} - \delta_{11} - \gamma_{12}.$$

Consequently,

$$\delta_{11} + \delta_{22} + \widehat{\tau}_1 - \widehat{\tau}_2 + \tau_3 = 2\delta_{12} - 2\gamma_{12}.$$

By the equality of the elements $\{\cdot\}_{22}$, we have

$$-\tau_3 = \delta_{22} - 2\delta_{12} + \delta_{11} - \gamma_{22}.$$

Summing the last two equalities yields

$$\widehat{\tau}_1 - \widehat{\tau}_2 = -2\gamma_{12} - \gamma_{22}.$$

Letting $\gamma_{22} = \widehat{\tau}_3$, we obtain

$$\gamma_{12} = (-\widehat{\tau}_1 + \widehat{\tau}_2 - \widehat{\tau}_3)/2.$$

Thus,

$$\begin{pmatrix} -\tau_1 & \dfrac{\tau_1 - \tau_2 + \tau_3}{2} \\ \bullet & -\tau_3 \end{pmatrix} = \left(\widehat{B}^\top L \widehat{B} - \Gamma\right),$$

where

$$\Gamma = \begin{pmatrix} \widehat{\tau}_1 & \dfrac{-\widehat{\tau}_1 + \widehat{\tau}_2 - \widehat{\tau}_3}{2} \\ \bullet & \widehat{\tau}_3 \end{pmatrix}.$$

The proof of Lemma 1 is complete.

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